Electric Circuit with Sinusoidal Voltage Source and Nonlinear Inductance
Analytical Model of Hysteresis Loop using Two Cubic Bezier Curves
Current and Magnetic-Flux-Linkage Waveforms
Harmonic Content of Current Waveform

Janusz Lipowski

Simple electric circuit is considered with sinusoidal voltage source and nonlinear inductance with magnetic core (with associated effects of magnetic saturation and hysteresis).

Inductances with saturable magnetic cores are associated with phenomena of magnetic saturation and hysteresis (the non-uniqueness of the magnetic flux for a given excitation current, with the value of magnetic flux depending on the value of the excitation current and its own previous state).

Magnetic saturation and hysteresis are two effects which define the shape of hysteresis loop.

Hysteresis loop is defined in a Cartesian plane between the end points (-1, -1) and (+1, +1). The first derivatives of the ascending and descending segments of a hysteresis loop are equal to zero at the end points.

Hysteresis loop can be defined by two cubic Bezier curves:
- the first one defines the ascending segment of hysteresis loop,
- the second one defines the descending segment of hysteresis loop.

Cubic Bezier curve

Cubic Bezier curve is defined by four control points: P1, P2, P3, P4, where P1, P4 are the end points, and P2, P3 are the handle points.

Cubic Bezier curve has two handles:
- the segment of straight line, of the length a, connecting points P1 and P2;
- the segment of straight line, of the length b, connecting points P3 and P4.

Cubic Bezier curve interpolates the end points P1, P4, and approximates the handle points P2, P3.

Cubic Bezier curve is bounded by the convex hull of its control points.
Equation defining cubic Bezier curve:

\[
P(t) = \sum_{i=1}^{4} P_i B_i(t) = \sum_{i=1}^{4} \begin{pmatrix} x_i \\ y_i \end{pmatrix} B_i(t)
\]

where \( B_1(t), B_2(t), B_3(t), B_4(t) \) are Bernstein polynomials:

\[
B_1(t) = (1-t)^3 \\
B_2(t) = 3t(1-t)^2 \\
B_3(t) = 3t^2(1-t) \\
B_4(t) = t^3
\]

\[
P(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4
\]

\( P(t) = [x(t), y(t)] \) is a point on the cubic Bezier curve, changing position with the change of value of variable \( t \) \((0 \leq t \leq 1)\)

Parametric equations defining cubic Bezier curve:

\[
x(t) = (1-t)^3 x_1 + 3t(1-t)^2 x_2 + 3t^2(1-t) x_3 + t^3 x_4 \\
y(t) = (1-t)^3 y_1 + 3t(1-t)^2 y_2 + 3t^2(1-t) y_3 + t^3 y_4
\]

**Equations defining hysteresis loop**

Parametric equations for cubic Bezier curve modelling ascending segment of a hysteresis loop:

\[
x(t) = (1-t)^3 x_1 + 3t(1-t)^2 x_2 + 3t^2(1-t) x_3 + t^3 x_4 \\
y(t) = (1-t)^3 y_1 + 3t(1-t)^2 y_2 + 3t^2(1-t) y_3 + t^3 y_4
\]

\((x_1, y_1) = (-1, -1)\) \\
\((x_2, y_2) = (-1+a, -1)\) \\
\((x_3, y_3) = (1-b, +1)\) \\
\((x_4, y_4) = (+1, +1)\)
Parametric equations for cubic Bezier curve modelling descending segment of a hysteresis loop:

\[ x(t) = (1-t)^3 x_1 + 3t(1-t)^2 x_2 + 3t^2(1-t) x_3 + t^3 x_4 \]
\[ y(t) = (1-t)^3 y_1 + 3t(1-t)^2 y_2 + 3t^2(1-t) y_3 + t^3 y_4 \]

\((x_1, y_1) = (-1, -1)\)
\((x_2, y_2) = (-1+b, -1)\)
\((x_3, y_3) = (1-a, +1)\)
\((x_4, y_4) = (+1, +1)\)

Parameters \(a\) and \(b\) \((0 < b < a)\) define the width and the slope of a hysteresis curve.

The presented model of hysteresis loop can be applied to the description of saturation and hysteresis of different physical nature in several areas of science where the saturation and hysteresis phenomena are encountered, for example: dielectric hysteresis, mechanical hysteresis, adsorption hysteresis, optical hysteresis, and so forth.

In this generalized model, as applied to the performance of the electric circuit with voltage source and nonlinear inductance, the major hysteresis loop is limited by points \((-1,-1)\) and \((1,1)\), and the normalized variables \(x(\alpha)\) and \(y(\alpha)\) defined as follows:

\[ U(\alpha) = \frac{d\psi(\alpha)}{dt} = -U_{\text{max}} \sin(\alpha) \quad \text{the sinusoidal voltage waveform} \quad -\pi \leq \alpha \leq \pi \]
\[ U_{\text{max}} = \omega_0 \psi_{\text{max}} \quad \text{the maximum value of voltage} \]

\[ I(\alpha) = \psi(\alpha) / I_{\text{max}} \quad \text{the non-sinusoidal current waveform} \quad -\pi \leq \alpha \leq \pi \]
\[ I_{\text{max}} \quad \text{the maximum value of current} \]

\[ \psi(\alpha) = \psi_{\text{max}} \cos(\alpha) \quad \text{the sinusoidal magnetic-flux-linkage waveform} \quad -\pi \leq \alpha \leq \pi \]
\[ \psi_{\text{max}} \quad \text{the maximum value of magnetic flux linkage} \]

\[ x(\alpha) = I(\alpha) / I_{\text{max}} \quad \text{the normalized independent variable waveform} \quad -\pi \leq \alpha \leq \pi \]
\[ y(\alpha) = \psi(\alpha) / \psi_{\text{max}} \quad \text{the normalized dependent variable waveform} \quad -\pi \leq \alpha \leq \pi \]

\[ \alpha = \omega_0 t \quad \text{angle (rd)} \]
\[ \omega_0 = 1 \text{ rd/sec} \quad \text{angular frequency (rd/sec)} \]
\[ t \quad \text{time (sec)} \]
The examples of hysteresis loops, and waveforms of current and magnetic flux linkage, generated with four different sets of parameters $a$ and $b$, are shown in Figures 1 to 4.

**Parametric equations defining cubic Bezier curve, shown in the form of cubic polynomial equations**

Parametric equations defining cubic Bezier curve:

\[
x(t) = (1-t)^3 x_1 + 3 t (1-t)^2 x_2 + 3 t^2 (1-t) x_3 + t^3 x_4
\]
\[
y(t) = (1-t)^3 y_1 + 3 t (1-t)^2 y_2 + 3 t^2 (1-t) y_3 + t^3 y_4
\]

Equivalent cubic polynomial equations, which can be used to determine the value of $t$, when $x(t)$, or $y(t)$ is given:

\[
a_1 t^3 + b_1 t^2 + c_1 t + d_1 = 0
\]
\[
a_2 t^3 + b_2 t^2 + c_2 t + d_2 = 0
\]

where

\[
a_1 = -x_1 + 3 x_2 - 3 x_3 + x_4
\]
\[
b_1 = 3 x_1 - 6 x_2 + 3 x_3
\]
\[
c_1 = -3 x_1 + 3 x_2
\]
\[
d_1 = x_1 - x(t)
\]
\[
a_2 = -y_1 + 3 y_2 - 3 y_3 + y_4
\]
\[
b_2 = 3 y_1 - 6 y_2 + 3 y_3
\]
\[
c_2 = -3 y_1 + 3 y_2
\]
\[
d_2 = y_1 - y(t)
\]

**Solution of a cubic polynomial equation:**

\[
a x^3 + b x^2 + c x + d = 0 \quad \text{or} \quad y^3 + 3 p y + 2 q = 0
\]

where

\[
x = y - b / (3 a)
\]
\[
p = (3 a c - b^2) / (9 a^2)
\]
\[
q = (-9 a b c + 27 a^2 d + 2 b^3) / (54 a^3)
\]
\[
delta = p^3 + q^2
\]
If $\delta \geq 0$  

Cardano’s solution

$$ u = (-q + \delta^{0.5})^{(1/3)} $$
$$ v = (-q - \delta^{0.5})^{(1/3)} $$

$$ x_1 = u + v - b / (3a) $$
$$ x_2 = -(u + v) / 2 - b / (3a) + 3^{0.5}(u - v)i / 2 $$
$$ x_3 = -(u + v) / 2 - b / (3a) - 3^{0.5}(u - v)i / 2 $$

If $\delta < 0$

$$ r = \text{sign}(q) |p|^{0.5} $$
$$ \phi = \arccos(q / r^3) $$

$$ x_1 = -2r \cos(\phi / 3) - b / (3a) $$
$$ x_2 = 2r \cos(\pi / 3 - \phi / 3) - b / (3a) $$
$$ x_3 = 2r \cos(\pi / 3 + \phi / 3) - b / (3a) $$

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Matlab script for solving a cubic polynomial equation: $a x^3 + b x^2 + c x + d = 0$

```matlab
a = ... ;
b = ... ;
c = ... ;
d = ... ;
p = (3*a*c - b^2) / (9*a^2);
q = (-9*a*b*c + 27*a^2*d + 2*b^3) / (54*a^3);
delta = p^3 + q^2
if delta >= 0
    u = nthroot(-q + sqrt(delta), 3);
    v = nthroot(-q - sqrt(delta), 3);
    x1 = u + v - b / (3*a);
    x2 = -(u + v) / 2 - b / (3*a) + i*3^{0.5}(u - v) / 2;
    x3 = -(u + v) / 2 - b / (3*a) - i*3^{0.5}(u - v) / 2;
else
    if q >= 0
        r = sqrt(abs(p));
    else
        r = -sqrt(abs(p));
    end
    phi = acos(q / r^3);
end
```
\begin{align*}
\text{x1} &= -2r \cos(\phi / 3) - b / (3a); \\
\text{x2} &= 2r \cos(\pi / 3 - \phi / 3) - b / (3a); \\
\text{x3} &= 2r \cos(\pi / 3 + \phi / 3) - b / (3a); \\
\end{align*}

\text{end}

\textbf{Computation of harmonic content of a periodic function f(x)}

The Fourier series expansion of an arbitrary periodic function \( f(x) = f(x + 2\pi) \), with a period \( T = 2\pi \), is defined as follows:

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \]

where Fourier coefficients \( a_0, a_n, b_n \ (n = 1, 2, 3, \ldots) \), are:

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \]

\textbf{Matlab script for computing harmonic content of a periodic function f(x)}

\text{(computing Fourier coefficients \( a_0, a_n, b_n, n=1, 2, 3, \ldots \))}

\% Computing harmonic content of the periodic function f(x)  
\% Computing Fourier Coefficients of the periodic function f(x)  
\% represented in the form of Fourier series

\text{nOh = \ldots; \quad \% number of harmonics to be computed}  
\text{alphamax = \ldots; \quad \% number of data points of f(x)}

\text{for j = 1 : noh}
\text{\quad for i = 1 : alphamax}
\text{\quad \quad xcos(i)=f(i)*cos(j*alpha(i));}  
\text{\quad \quad xsin(i)=f(i)*sin(j*alpha(i));}  
\text{\quad end}
\text{\quad aa(j) = 1/pi*trapz(alpha,xcos);}  
\text{\quad bb(j) = 1/pi*trapz(alpha,xsin);}  
\text{\quad end}
Computation of harmonic content of current waveform

The harmonic content of current waveform $x(\alpha)$, shown in the form of Fourier series, can be determined by the values of corresponding Fourier coefficients. The harmonic content of current waveform $x(\alpha)$ has been determined by computing Fourier coefficients for the first seven harmonics of the current waveform, for four different sets of values of coefficients $a$ and $b$ (determining the width and slope of hysteresis loop). The computed Fourier coefficients $a_i$ and $b_i$, $i = 1, 3, 5, 7$, for the first seven harmonics of the current waveform (shown in Figures: 1b, 2b, 3b, 4b), are as follows.

```
a = 1.5,  b = 1  

<table>
<thead>
<tr>
<th>Harmonic</th>
<th>$a_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6757</td>
<td>-0.1910</td>
</tr>
<tr>
<td>3</td>
<td>0.1552</td>
<td>-0.0041</td>
</tr>
<tr>
<td>5</td>
<td>0.0555</td>
<td>-0.0009</td>
</tr>
<tr>
<td>7</td>
<td>0.0283</td>
<td>-0.0003</td>
</tr>
</tbody>
</table>

$x(\alpha) = a_1 \cos(\alpha) + b_1 \sin(\alpha) + a_3 \cos(3\alpha) + b_3 \sin(3\alpha) + a_5 \cos(5\alpha) + b_5 \sin(5\alpha) + a_7 \cos(7\alpha) + b_7 \sin(7\alpha)$
```

```
a = 2,  b = 1.5  

<table>
<thead>
<tr>
<th>Harmonic</th>
<th>$a_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5460</td>
<td>-0.1910</td>
</tr>
<tr>
<td>3</td>
<td>0.2172</td>
<td>-0.0041</td>
</tr>
<tr>
<td>5</td>
<td>0.0777</td>
<td>-0.0009</td>
</tr>
<tr>
<td>7</td>
<td>0.0396</td>
<td>-0.0003</td>
</tr>
</tbody>
</table>

$x(\alpha) = a_1 \cos(\alpha) + b_1 \sin(\alpha) + a_3 \cos(3\alpha) + b_3 \sin(3\alpha) + a_5 \cos(5\alpha) + b_5 \sin(5\alpha) + a_7 \cos(7\alpha) + b_7 \sin(7\alpha)$
```

```
a = 2,  b = 1  

<table>
<thead>
<tr>
<th>Harmonic</th>
<th>$a_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6109</td>
<td>-0.3820</td>
</tr>
<tr>
<td>3</td>
<td>0.1862</td>
<td>-0.0083</td>
</tr>
<tr>
<td>5</td>
<td>0.0666</td>
<td>-0.0017</td>
</tr>
<tr>
<td>7</td>
<td>0.0340</td>
<td>-0.0006</td>
</tr>
</tbody>
</table>

$x(\alpha) = a_1 \cos(\alpha) + b_1 \sin(\alpha) + a_3 \cos(3\alpha) + b_3 \sin(3\alpha) + a_5 \cos(5\alpha) + b_5 \sin(5\alpha) + a_7 \cos(7\alpha) + b_7 \sin(7\alpha)$
```

```
a = 2.5,  b = 1  

<table>
<thead>
<tr>
<th>Harmonic</th>
<th>$a_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5460</td>
<td>-0.5730</td>
</tr>
<tr>
<td>3</td>
<td>0.2172</td>
<td>-0.0124</td>
</tr>
<tr>
<td>5</td>
<td>0.0777</td>
<td>-0.0026</td>
</tr>
<tr>
<td>7</td>
<td>0.0396</td>
<td>-0.0009</td>
</tr>
</tbody>
</table>

$x(\alpha) = a_1 \cos(\alpha) + b_1 \sin(\alpha) + a_3 \cos(3\alpha) + b_3 \sin(3\alpha) + a_5 \cos(5\alpha) + b_5 \sin(5\alpha) + a_7 \cos(7\alpha) + b_7 \sin(7\alpha)$
```
Figure 1a - Hysteresis Loop, $a=1.5$, $b=1$

Figure 1b - Magnetic-Flux-Linkage and Current Waveforms, $a=1.5$, $b=1$
Figure 2a - Hysteresis Loop, $a=2$, $b=1.5$

Figure 2b - Magnetic-Flux-Linkage and Current Waveforms, $a=2$, $b=1.5$
Figure 3a - Hysteresis Loop, a=2, b=1

Figure 3b - Magnetic-Flux-Linkage and Current Waveforms, a=2, b=1
Figure 4a - Hysteresis Loop, a=2.5, b=1

Figure 4b - Magnetic-Flux-Linkage and Current Waveforms, a=2.5, b=1
% Electric Circuit with Sinusoidal Voltage Source
% and Nonlinear Inductance
% Hysteresis Loop Modelling with Cubic Bezier Curves
% Non-sinusoidal Current and Sinusoidal Magnetic-Flux-Linkage Waveforms
% Harmonic Content of Non-sinusoidal Current Waveform
%---------------------------------------------------------------
% Parameters defining the width and the slope of hysteresis loop: a, b
% 0 < b < a

% Ascending segment of hysteresis loop
% [xx(1), yy(1)] = [-1, -1]
% [xx(2), yy(2)] = [-1+a, -1]
% [xx(3), yy(3)] = [1-b, 1]
% [xx(4), yy(4)] = [1, 1]

% Descending segment of hysteresis loop
% [xx(5), yy(5)] = [-1, -1]
% [xx(6), yy(6)] = [-1+b, -1]
% [xx(7), yy(7)] = [1-a, 1]
% [xx(8), yy(8)] = [1, 1]

% Adjustable parameters
a=2.5;
b=1;
delta = pi/100;

noh = 7;   % Number of harmonics to be computed
alphamiddle = 101;
alphamax = 201;

% Fixed (unadjustable) part of program
M = [-1, -1; -1+a, -1; 1-b, 1; 1, 1; -1, -1; -1+b, -1; 1-a, 1; 1, 1];

xx = M(:, 1);
yy = M(:, 2);
t_array = 0:0.05:1;
for i = 1:length(t_array)
    t = t_array(i);
    fxa(i) = (1-t)^3*xx(1)+3*t*(1-t)^2*xx(2)+3*t^2*(1-t)*xx(3)+t^3*xx(4);
    fya(i) = (1-t)^3*yy(1)+3*t*(1-t)^2*yy(2)+3*t^2*(1-t)*yy(3)+t^3*yy(4);
    fxb(i) = (1-t)^3*xx(5)+3*t*(1-t)^2*xx(6)+3*t^2*(1-t)*xx(7)+t^3*xx(8);
    fyb(i) = (1-t)^3*yy(5)+3*t*(1-t)^2*yy(6)+3*t^2*(1-t)*yy(7)+t^3*yy(8);
end

plot(fxa, fya);
hold on
plot(fxb, fyb);
% Computing and plotting waveforms of current x(alpha)=I(alpha)/Imax, % and magnetic-flux-linkage y(alpha)=Psi(alpha)/Psimax % Current waveform is non-sinusoidal % Magnetic-flux-linkage waveform is sinusoidal % (due to sinusoidal voltage waveform)

for i = 1 : alphamax
    alpha(i) = - pi + (i-1)*delta;
end

for i = 1 : alphamax
    y(i) = cos(alpha(i));
end

% Ascending segment of hysteresis loop
for i = 1 : alphamiddle
    a1 = - yy(1) + 3*yy(2) - 3*yy(3) + yy(4);
    b1 = 3*yy(1) - 6*yy(2) + 3*yy(3);
    c1 = -3*yy(1) + 3*yy(2);
    d1 = yy(1) - y(i);
    t = ff(a1, b1, c1, d1);
    x(i) = (1-t)^3*xx(1) + 3*t*(1-t)^2*xx(2) + 3*t^2*(1-t)*xx(3) + t^3*xx(4);
end

% Descending segment of hysteresis loop
for i = alphamiddle : alphamax
    a2 = - yy(5) + 3*yy(6) - 3*yy(7) + yy(8);
    b2 = 3*yy(5) - 6*yy(6) + 3*yy(7);
    c2 = -3*yy(5) + 3*yy(6);
    d2 = yy(5) - y(i);
    t = ff(a2, b2, c2, d2);
    x(i) = (1-t)^3*xx(5) + 3*t*(1-t)^2*xx(6) + 3*t^2*(1-t)*xx(7) + t^3*xx(8);
end

figure;
plot (alpha, x)
hold on
plot (alpha, y)

% Computing harmonic content of the non-sinusoidal current waveform % Computing Fourier Coefficients of the non-sinusoidal current waveform % represented in the form of Fourier series

for j = 1 : noh
    for i = 1 : alphamax
        xcos(i)=x(i)*cos(j*alpha(i));
        xsin(i)=x(i)*sin(j*alpha(i));
    end
    aa(j) = 1/pi*trapz(alpha,xcos);
    bb(j) = 1/pi*trapz(alpha,xsin);
end
cc = [aa; bb]

fileID = fopen('coefficients_a_b.txt','w');
fprintf(fileID, '%6s %9s
', 'aa', 'bb');
fprintf(fileID, '%8.4f; %8.4f
', cc);
fclose(fileID);

% Function for the solution of a cubic polynomial equation

function xx = ff(a, b, c, d)
p = (3*a*c - b^2)/(9*a^2);
q = (-9*a*b*c + 27*a^2*d + 2*b^3)/(54*a^3);
delta = p^3 + q^2

if delta >= 0
    u = nthroot(-q + sqrt(delta), 3);
v = nthroot(-q - sqrt(delta), 3);
x1 = u + v - b/(3*a);
x2 = -(u + v)/2 - b/(3*a) + i*3^0.5*(u - v)/2;
x3 = -(u + v)/2 - b/(3*a) - i*3^0.5*(u - v)/2;
else
    if q >= 0
        r = sqrt(abs(p));
    else
        r = -sqrt(abs(p));
    end
    phi = acos(q/r^3);
x1 = -2*r*cos(phi/3) - b/(3*a);
x2 = 2*r*cos(pi/3 - phi/3) - b/(3*a);
x3 = 2*r*cos(pi/3 + phi/3) - b/(3*a);
end

if (imag(x1) == 0) && (x1 >= 0) && (x1 <= 1)
    xx = x1;
elseif (imag(x2) == 0) && (x2 >= 0) && (x2 <= 1)
    xx = x2;
elseif (imag(x3) == 0) && (x3 >= 0) && (x3 <= 1)
    xx = x3;
else
    xx = 9999999;
end